THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Homework 2
Due Date: 26th September 2024

Compulsory Part

1. When $A = \{a\}$ is a singleton, show that the free group $F(A)$ is isomorphic to the infinite cyclic group $\mathbb{Z}$.

2. Verify that $\mathbb{Z}^{\oplus A} := \{f : A \to \mathbb{Z} : f(a) \neq 0$ for only finitely many $a \in A\}$ is indeed an abelian group, for any given set A.

3. Show that a finitely generated abelian group can be presented as a quotient of $\mathbb{Z}^{\oplus n}$ for some positive integer n.

4. Let G be a group. For any $g \in G$, the map $i_g : G \to G$ defined by $i_g(a) = gag^{-1}$ for any $a \in G$ is an automorphism of G, which is called an **inner automorphism** of G. Prove that the set $\text{Inn}(G)$ of inner automorphisms of G is a normal subgroup of the automorphism group $\text{Aut}(G)$ of G.

   [*Warning:* Be sure to show that the inner automorphisms do form a subgroup.]

5. Show that an intersection of normal subgroups of a group G is again a normal subgroup of G.

6. Let G be a group containing at least one subgroup of a fixed finite order s. Show that the intersection of all subgroups of G of order s is a normal subgroup of G.

   [*Hint:* Use the fact that if H has order s, then so does $x^{-1}Hx$ for all $x \in G$.]

Optional Part

1. Let G be a finite group with $|G|$ odd. Show that the equation $x^2 = a$, where x is the indeterminate and a is any element in G, always has a solution. (In other words, every element in G is a square.)

2. Generalizing the above question: If G is a finite group of order n and k is an integer relatively prime to n, show that the map $G \to G, a \mapsto a^k$ is surjective.

3. Prove that every finite group is finitely presented.

4. Prove that $(\mathbb{Q}_{>0}, \cdot)$ is a free abelian group, meaning that it is isomorphic to $\mathbb{Z}^{\oplus A}$ for some set A.

   [*Hint*: Use the fundamental theorem of arithemetic, i.e., every positive integer can be uniquely factorized as a product of primes.]

5. We have learnt that a presentation of the dihedral group D_n is given by $(a, b \mid a^2, b^n)$

   Let a, b be distinct elements of order 2 in a group G. Suppose that ab has finite order $n \geq 3$. Prove that the subgroup $\langle a, b\rangle$ generated by a and b is isomorphic to the dihedral group D_n (which has $2n$ elements).

6. Let $G = \mathbb{Z}^{\oplus \mathbb{N}}$. Prove that $G \times G \cong G$ (as abelian groups).

7. Prove that $(\mathbb{Q}, +)$ is not a free abelian group.

8. Show that if a finite group G has exactly one subgroup H of a given order, then H is a normal subgroup of G.

9. Show that the set of all $g \in G$ such that the inner automorphism $i_g : G \to G$ is the identity inner automorphism i_e is a normal subgroup of a group G.